# Biophysics 702 Homework 2: Modeling and Simulation

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To model population growth, one might initially use model such as:

$$\frac{dN}{dt} = \gamma N \tag{1}$$

where $\gamma$ reflects the rate of growth. This simple differential model of population growth has the rate of change in the population directly proportional to the size of the population. Clearly such growth cannot be sustained, as the population will expand exponentially and rapidly reach the "carrying capacity" for the environment, a population at which the population should be essentially stable, and beyond which population growth should be negative.

We can expand the simplistic model to take into account an environmental carrying capacity by multiplying in a term that is proportional to the difference between the current population and the carrying capacity. Such a model might be represented by the differential equation:

$$\frac{dN}{dt} = \gamma N(Q - N) \tag{2}$$

where $Q$ is the maximum continuously sustainable population in the environment. Dividing by $Q$, we get:

$$\frac{dx}{dt} = rx(1 - x) \tag{3}$$

$x$ is the proportion of the carrying capacity currently occupied, and $r$ is a constant proportional to $Q$ and $\gamma$. This differential equation has the solution $x(t) = \frac{1}{1+Ce^{-rt}}$, where $C$ comes from the inital conditions.

If we want to model this system in a computer, we might choose to discretize the differential equation and simulate the behavior of the system at discrete timepoints. Discretizing (3) in time, we arrive at the equation:

$$\frac{x_{t+h} - x_t}{h} = rx_t(1 - x_t) \tag{4}$$

$$x_{t+h} = hrx_t(1 - x_t) + x_t \quad (5)$$

$$x_{t+h} = hrx_t - hr{x_t}^2 + x_t \quad (6)$$

$$x_{t+h} = hrx_t + x_t - hr{x_t}^2 \quad (7)$$

$$x_{t+h} = hrx_t((1 + \frac{1}{hr}) - x_t) \quad (8)$$

Dividing by $1 + \frac{1}{hr}$ produces:

$$\tilde{x}_{t+h} = \tilde{r}\tilde{x}_t(1 - \tilde{x}_t)^1 \quad (9)$$

where $\tilde{x} = \frac{x}{1+\frac{1}{hr}}$ and $\tilde{r} = hr + 1$.

Questions:

1. Does the discretization of (3) accurately model the behavior of (3)? Under what circumstances does it fail? Particularly, test the behavior over a few hundred timepoints with several values of $\tilde{r}$ in the neighborhoods of 2, 3, 3.59, and 4. A low value of $\tilde{x}_t$ is typically a good starting place: $\tilde{x}_t$ =< .00001 works well, but interesting behaviors should result with any initial values for $\tilde{x}_t$ other than 0 and 1.

2. What is the failure mode, and why does it occur?

3. Is there a way to circumvent the failure?

A Perl program that implements a simple graph of this function is downloadable from the web-page as `test_disc.pl`. You will need to edit the top of the file to indicate whether it should produce a PNG graph of the output, or a tab-delimited list of $(t, \tilde{x}_t)$ coordinates. You will also need to edit it for each value of $\tilde{r}$ that you want to test. If you choose to write the output as a PNG graph, in this representation, $(0, 0)$ is in the upper left, positive $t$ is to the right, and positive $\tilde{x}$ is down.

The output is written to STDOUT, so you will want to capture it into either a text file (foo.txt), or a PNG file (foo.png) depending on the output type you've chosen. `test_disc.pl > foo.png` or `test_disc.pl > foo.txt` will capture the results for you.

[1] Interestingly, it turns out that the discrete form (9) of (3) looks essentially identical to the continuous differential equation from which it is derived - this is not universally, or often the case.